2017-2018 年度美国“大联盟”(Math League)思维探索第一阶段活动

（十、十一、十二年级）

（活动日期：2017 年 11 月 26 日，答题时间：90 分钟，总分 300 分）

学生诚信协议：答题期间，我确定没有就所涉及的问题或结论，与任何人、用任何方式交流或讨论，我确定我所填写的答案均为我个人独立完成的成果，否则愿接受本次成绩无效的处罚。

填空题(每小题 10 分，答对加 10 分，答错不扣分，共 300 分。)

1. Each pirate wants his own treasure chest, but there is 1 more pirate than there are treasure chests. If the pirates would agree to pair up so each pirate shares a treasure chest with another pirate, then 1 treasure chest would not be assigned to any pirate. How many treasure chests are there?

Answer: ________________.

2. If m and n are positive integers that satisfy $\sqrt{m}+\sqrt{n}=10$, what is the greatest possible value of $m+n$?

Answer: ________________.

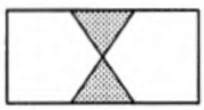
3. There are an infinite number of points with positive coordinates (x,y) the sum of whose coordinates is the square of an integer. Among all such points (x,y), which one satisfies $y=2x$ and has x as small as possible?

Answer: ________________.

4. As shown, a small square is inscribed in one of the triangles formed when both diagonals of a larger square are drawn. If the area of the larger square is 144, what is the area of the smaller square?

Answer: ________________.

5. Trisection points on opposite sides of a rectangle are joined, as shown. If the area of the shaded region is 2018, what is the area of the rectangle?



Answer: ________________.

6. A unit fraction is a fraction whose numerator is 1 and whose denominator is a positive integer. What is the largest rational number that can be written as the sum of 3 different unit fractions?

Answer: ________________.

7. What is the greatest possible perimeter of a rectangle whose length and width are different prime numbers, each less than 120?

Answer: ________________.

8. Mom, Dad, and I each write a positive integer. My number is least and Dad's is greatest. The average of all 3 numbers is 20. The average of the 2 smallest numbers is 8. If Dad's number is d and if my number is m, what is the greatest possible value of $d - m$?

Answer: ________________.

9. If 8 different integers are chosen at random from the first 15 positive integers, what is the probability that an additional number chosen at random from the remaining 7 positive integers is smaller than every one of the 8 originally chosen positive integers?

Answer: ________________.

10. What sequence of 5 positive integers has these three properties:
   1) All but one of the numbers is a multiple of 5.
   2) Every number after the first is 1 more than the sum of all the preceding numbers.
   3) The first number is as small as possible.

Answer: ________________.

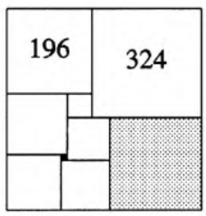
11. Three beavers (one not shown) take turns biting a tree until it falls. The second beaver is twice as likely as the first to make the tree fall. The third is twice as likely as the second to make the tree fall. What is the probability that a bite taken by the third beaver causes the tree to fall?

Answer: ________________.

12. What is the ratio, larger to smaller, of a rectangle's dimensions if half of the rectangle is similar to the original rectangle?

Answer: ________________.

13. A rectangle is partitioned into 9 different squares, as shown at the right. The area of the smallest square, shown fully darkened, is 1. Two other squares have areas of 196 and 324, as shown. What is the area of the shaded square?



Answer: ________________.

14. When the square of an eight-digit integer is subtracted from the square of a different eight-digit integer, the difference will sometimes have eight identical even digits. What are both possible values of the repeated digit in such a situation?

Answer: ________________.

15. If the perimeter of an isosceles triangle with integral sides is 2017, how many different lengths are possible for the legs?

Answer: ________________.

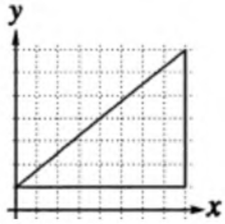
16. What are all ordered triples of positive primes (p,q,r) which satisfy $p^q + 1 = r$?

Answer: ________________.

17. The reflection of (6,3) across the line $x = 4$ is (2,3). If $m \neq 4$, what is the reflection of (m,n) across the line $x = 4$?

Answer: ________________.

18. The vertices of a triangle are (8,7), (0,1), and (8,1). What are the coordinates of all points inside this triangle that have integral coordinates and lie on the bisector of the smallest angle of the triangle?

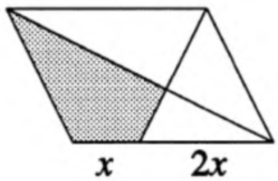


Answer: ________________.

19. In a regular 10-sided polygon, two pairs of different vertices (four different vertices altogether) are chosen at random, so that all points chosen are distinct from each other. What is the probability that the line segments determined by each pair of points do *not* intersect?

Answer: ________________.

20. A line segment is drawn from the upper right vertex of a parallelogram, as shown, dividing the opposite side into segments with lengths in a 2:1 ratio. If the area of the parallelogram is 90, what is the area of the shaded region?



Answer: ________________.

21. If $0 < a \leq b \leq 1$, what is the maximum value of $ab^2 - a^2b$?

Answer: ________________.

22. What are all ordered pairs of integers (x,y) that satisfy $5x^3 + 2xy - 23 = 0$?

Answer: ________________.

23. If two altitudes of a triangle have lengths 10 and 15, what is the smallest integer that could be the length of the third altitude?

Answer: ________________.

24. If h is the number of heads obtained when 4 fair coins are each tossed once, what is the expected (average) value of h^2?

Answer: ________________.

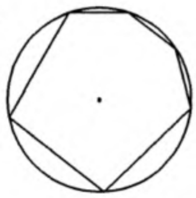
25. What is the largest integer N for which $7x + 11y = N$ has no solution in non-negative integers (x,y)?

Answer: ________________.

26. There are only two six-digit integers n greater than 100 000 for which n^2 has n as its final six digits (or, equivalently, for which $n^2 - n$ is divisible by 10^6). One of the integers is 890 625. What is the other?

Answer: ________________.

27. A hexagon is inscribed in a circle as shown. If lengths of three sides of the hexagon are each 1 and the lengths of the other three sides are each 2, what is the area of this hexagon? Write your answer in its exact format or round to the nearest tenth.



Answer: _________________.

28. If x is a number chosen uniformly at random between 0 and 1, what is the probability that the greatest integer $\leq \log_2\left(\frac{1}{x}\right)$ is odd?

Answer: _________________.

29. In the interval $-1 < x < 1$, $\sin\theta$ is one root of $x^4 - 4x^3 + 2x^2 - 4x + 1 = 0$. In that same interval, for what ordered pair of integers (a,b) is $\cos 2\theta$ one root of $x^2 + ax + b = 0$?

Answer: _________________.

30. Let $P(x) = 2x^{10} + 3x^9 + 4x + 9$. If z is a non-real solution of $z^3 = 1$, what is the numerical value of $P\left(\frac{1}{z}\right) + P\left(\frac{1}{z^2}\right) + P\left(\frac{1}{z^3}\right)$?

Answer: _________________.

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十、十一、十二年级试卷答案

题号	1	2	3	4	5	6	7	8	9	10
答案	3	82	$\left(\frac{1}{3},\frac{2}{3}\right)$	16	12108	$\frac{11}{6}$	444	43	$\frac{1}{9}$	4,5,10, 20,40
题号	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
答案	$\frac{4}{7}$	$\sqrt{2}$	225	4, 8	504	(2,2,5)	$(8-m,n)$	(3,2), (6,3)	$\frac{2}{3}$	33
题号	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
答案	$\frac{1}{4}$	(1,9), (-1,-14), (23,-1322), (-23,-1323)	7	5	59	109376	$13\sqrt{3}/4$ or 5.6	$\frac{1}{3}$	(26, -23)	36